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ANALYSIS OF UNBALANCED ANGLE-PLY RECTANGULAR PLATES†

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Abstract—Unbalanced angle-ply laminated rectangular plates under transverse normal loading are analyzed. By expanding the load and the displacements in single Fourier series the governing partial differential equations are reduced to ordinary differential equations for simple support boundary conditions on two opposite edges and arbitrary boundary conditions on the other two edges. The closed form solution for the ordinary differential equations is obtained for 45° angle-ply laminates. Several numerical examples are solved and their results are presented in graphs. The degree of coupling between stretching and bending depends on the transverse loading, the edge boundary conditions and the aspect ratio of the plate, as well as on the degree of anisotropy of the layers, the angle of orientation and the number of plies in the composite. The type of in-plane boundary condition (movable or immovable) has a significant effect on bending for simply supported plates but not for clamped plates. The Fourier series method discussed in this paper can be used with the method of superposition to solve problems having other edge conditions, as well as transverse loading conditions.

NOTATION

A_{ij}	in-plane stiffness coefficients of plate
B_{ij}	coupling stiffness coefficients of plate
C_{ij}	reduced stiffness coefficients of a layer
D_{ij}	flexural stiffness coefficients of plate
E	Young's modulus
G	Shear modulus
L	dimension of square plate ($a = b = L$)
M	component of stress couple resultant
N	component of in-plane stress resultant
Q	component of transverse shear stress resultant
a	plate length along x direction
b	plate width along y direction
h	plate thickness along z direction
n	number of plies
p	transverse normal load at $z = h/2$
u, v, w	displacement components in x, y, z directions
u^0, v^0, w^0	middle surface displacement components
x, y, z	Cartesian coordinates
κ	component of curvature
ε	component of strain
ε^0	component of middle surface strain
τ	component of stress
ν	Poisson's ratio
θ	orientation angle of composite
$(\ )_{,i}$	partial differentiation with respect to $i = x, y, z$

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INTRODUCTION

THE theory of anisotropic laminated plates has been well established [1–6]. In general, the plate heterogeneity introduces a coupling phenomenon [2–6] between bending and stretching even in the case of small deflections based on Kirchhoff's hypothesis. To date, analytical solutions for coupled angle-ply laminates with general boundary conditions are limited. Reissner and Stavsky [2] obtained solutions for an infinite coupled plate under transverse loading and under a combination of uniform tension and uniform bending moment. Recently, Whitney and Leissa [7–9] obtained double Fourier series solutions for the transverse loading of coupled rectangular plates.

In this paper, a single Fourier series displacement formulation is used to obtain an analytical solution for unbalanced angle-ply rectangular plates with simple support boundary conditions on two opposite edges and general boundary conditions on the other two edges. Unbalanced angle-ply plates consist of an even number of bonded layers of elastically orthotropic sheets of equal thickness. The odd numbered plies are oriented with an angle $+\theta$ and the even numbered plies with an angle $-\theta$ to the plate axes.

FORMULATION OF PROBLEM

Let us consider a thin elastic laminated plate of thickness h, referred to an x, y, z system of Cartesian coordinates. The lower and upper surfaces of the plate are $z = \pm h/2$. The faces of the plate are assumed to be free of shear stresses but subjected to transverse normal stress, as follows

$$\tau_{xz}|_{z=\pm h/2} = 0, \qquad \tau_{yz}|_{z=\pm h/2} = 0 \tag{1}$$

$$\tau_{z}|_{z=-h/2} = 0, \qquad \tau_{z}|_{z=+h/2} = p(x, y). \tag{2}$$

Linear plate theory based on Kirchhoff's hypothesis leads to the following displacement, strain and curvature relations

$$u = u^0(x, y) - zw^0_{,x}, \qquad v = v^0(x, y) - zw^0_{,y}, \qquad w = w^0(x, y) \tag{3}$$

$$(\varepsilon_x, \varepsilon_y, \varepsilon_{xy}) = (\varepsilon^0_x, \varepsilon^0_y, \varepsilon^0_{xy}) + z(\kappa_x, \kappa_y, \kappa_{xy}) \tag{4}$$

$$\varepsilon^0_x = u^0_{,x}, \qquad \varepsilon^0_y = v^0_{,y}, \qquad \varepsilon^0_{xy} = u^0_{,y} + v^0_{,x} \tag{5}$$

$$\kappa_x = -w^0_{,xx}, \qquad \kappa_y = -w^0_{,yy}, \qquad \kappa_{xy} = -2w^0_{,xy}. \tag{6}$$

Thus, normals to the undeformed middle surface are assumed to remain normal to the deformed middle surface and undergo no extension. Also transverse shear deformation is neglected.†

Hooke's law for the kth homogeneous orthotropic sheet is given by the relation

$$\begin{Bmatrix} \tau_x \\ \tau_y \\ \tau_{xy} \end{Bmatrix} = \begin{bmatrix} C^{(k)}_{11} & C^{(k)}_{12} & C^{(k)}_{16} \\ C^{(k)}_{12} & C^{(k)}_{22} & C^{(k)}_{26} \\ C^{(k)}_{16} & C^{(k)}_{26} & C^{(k)}_{66} \end{bmatrix} \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \varepsilon_{xy} \end{Bmatrix} \tag{7}$$

† See the results of [10, 11] for the cylindrical bending of composite laminates with shear deformation. The effect of shear deformation at high span-to-depth ratio is very small.

where the effect of transverse normal stress is neglected. The $C_{ij}^{(k)}$ quantities are reduced stiffness coefficients as discussed by Tsai and Pagano [12] such that for the angle-ply construction specified

$$C_{ij}^{(k)} = C_{ij}^{(k+1)} \quad \text{except} \quad C_{16}^{(k)} = -C_{16}^{(k+1)}, \qquad C_{26}^{(k)} = -C_{26}^{(k+1)}. \tag{8}$$

In-plane stress resultants, transverse shear stress resultants and stress couple resultants are defined as follows

$$(N_x, N_y, N_{xy}) = \int_{-h/2}^{+h/2} (\tau_x, \tau_y, \tau_{xy})\, dz \tag{9}$$

$$(Q_x, Q_y) = \int_{-h/2}^{h/2} (\tau_{xz}, \tau_{yz})\, dz \tag{10}$$

$$(M_x, M_y, M_{xy}) = \int_{-h/2}^{h/2} (\tau_x, \tau_y, \tau_{xy}) z\, dz. \tag{11}$$

The two-dimensional coupled linear theory of anisotropic laminated plates is deduced from the equilibrium equations of three-dimensional elasticity

$$\tau_{x,x} + \tau_{xy,y} + \tau_{xz,z} = 0 \tag{12}$$

$$\tau_{xy,x} + \tau_{y,y} + \tau_{yz,z} = 0 \tag{13}$$

$$\tau_{xz,x} + \tau_{yz,y} + \tau_{z,z} = 0 \tag{14}$$

where body forces are not considered. These equations are converted to plate-stress equilibrium equations by the method of Boussinesq. First equations (12)–(14) are integrated over the plate thickness and then equations (12) and (13) are multiplied by z and integrated over the thickness. The following equilibrium equations [5] are obtained

$$N_{x,x} + N_{xy,y} = 0 \tag{15}$$

$$N_{xy,x} + N_{y,y} = 0 \tag{16}$$

$$Q_{x,x} + Q_{y,y} + p = 0 \tag{17}$$

$$M_{x,x} + M_{xy,y} - Q_x = 0 \tag{18}$$

$$M_{xy,x} + M_{y,y} - Q_y = 0. \tag{19}$$

Combining equations (17)–(19) to eliminate Q_x and Q_y gives the result

$$M_{x,xx} + 2M_{xy,xy} + M_{y,yy} + p = 0. \tag{20}$$

In order to obtain plate stress–strain relations expression (4) is introduced into (7) and the results are integrated according to definitions (9)–(11) and conditions (8) to give

$$\begin{Bmatrix} N_x \\ N_y \\ N_{xy} \\ M_x \\ M_y \\ M_{xy} \end{Bmatrix} = \begin{bmatrix} A_{11} & A_{12} & 0 & 0 & 0 & B_{16} \\ A_{12} & A_{22} & 0 & 0 & 0 & B_{26} \\ 0 & 0 & A_{66} & B_{16} & B_{26} & 0 \\ 0 & 0 & B_{16} & D_{11} & D_{12} & 0 \\ 0 & 0 & B_{26} & D_{12} & D_{22} & 0 \\ B_{16} & B_{26} & 0 & 0 & 0 & D_{66} \end{bmatrix} \begin{Bmatrix} \varepsilon_x^0 \\ \varepsilon_y^0 \\ \varepsilon_{xy}^0 \\ \kappa_x \\ \kappa_y \\ \kappa_{xy} \end{Bmatrix} \tag{21}$$

where the constants A_{ij}, B_{ij} and D_{ij} are defined by the following integrals for a n-layer laminated plate

$$(A_{ij}, B_{ij}, D_{ij}) = \sum_{k=1}^{n} \int_{h_{k-1}}^{h_k} C_{ij}^{(k)}(1, z, z^2)\,dz \qquad (i,j = 1, 2, 6). \tag{22}$$

The number of layers n is even and the thickness of each layer is h/n. The coupling effect is introduced through the quantities B_{16} and B_{26}.

Substituting the plate Hooke's law relation (21) into the equilibrium equations (15), (16) and (20) and using the kinematic equations (5) and (6), the following expressions for equilibrium in terms of displacements are obtained

$$A_{11}u^0_{,xx} + A_{66}u^0_{,yy} + (A_{12}+A_{66})v^0_{,xy} - 3B_{16}w^0_{,xxy} - B_{26}w^0_{,yyy} = 0 \tag{23}$$

$$(A_{12}+A_{66})u^0_{,xy} + A_{66}v^0_{,xx} + A_{22}v^0_{,yy} - B_{16}w^0_{,xxx} - 3B_{26}w^0_{,xyy} = 0 \tag{24}$$

$$D_{11}w^0_{,xxxx} + 2(D_{12}+2D_{66})w^0_{,xxyy} + D_{22}w^0_{,yyyy} - B_{16}(v^0_{,xxx} + 3u^0_{,xxy}) - B_{26}(u^0_{,yyy} + 3v^0_{,xyy}) = p. \tag{25}$$

Consider a rectangular angle-ply plate with two opposite edges simply supported. The simple supports are smooth pins which do not allow normal displacements, but do allow lateral contraction (tangential displacements). Thus, the boundary conditions at $y = 0$ and $y = b$ are given as

$$w^0 = M_y = v^0 = N_{xy} = 0. \tag{26}$$

The remaining boundary conditions at $x = 0$ and $x = a$ are arbitrary.

ANALYTICAL SOLUTION

Let us expand the transverse normal load $p(x, y)$ into a half range Fourier sine series in one variable y

$$p = \sum_{m=1}^{\infty} p^{(m)}(x) \sin \frac{m\pi y}{b}. \tag{27}$$

Solutions are of the form

$$u^0 = \sum_{m=1}^{\infty} u^{(m)}(x)\cos\frac{m\pi y}{b} \quad (28)$$

$$v^0 = \sum_{m=1}^{\infty} v^{(m)}(x)\sin\frac{m\pi y}{b} \quad (29)$$

$$w^0 = \sum_{m=1}^{\infty} w^{(m)}(x)\sin\frac{m\pi y}{b}. \quad (30)$$

The boundary conditions (26) are identically satisfied by the in-plane displacements (28) and (29) and the transverse deflection (30). Substituting equations (27)–(30) into the governing equations (23)–(25) gives an eight-order system of ordinary differential equations for the variables $u^{(m)}(x)$, $v^{(m)}(x)$ and $w^{(m)}(x)$ in the form

$$A_{11}u^{(m)}_{,xx} - A_{66}\left(\frac{m\pi}{b}\right)^2 u^{(m)} + (A_{12}+A_{66})\left(\frac{m\pi}{b}\right)v^{(m)}_{,x} - 3B_{16}\left(\frac{m\pi}{b}\right)w^{(m)}_{,xx} + B_{26}\left(\frac{m\pi}{b}\right)^3 w^{(m)} = 0 \quad (31)$$

$$-(A_{12}+A_{66})\left(\frac{m\pi}{b}\right)u^{(m)}_{,x} + A_{66}v^{(m)}_{,xx} - A_{22}\left(\frac{m\pi}{b}\right)^2 v^{(m)} - B_{16}w^{(m)}_{,xxx} + 3B_{26}\left(\frac{m\pi}{b}\right)^2 w^{(m)}_{,x} = 0 \quad (32)$$

$$D_{11}w^{(m)}_{,xxxx} - 2(D_{12}+2D_{66})\left(\frac{m\pi}{b}\right)^2 w^{(m)}_{,xx} + D_{22}\left(\frac{m\pi}{b}\right)^4 w^{(m)} - B_{16}\left[v^{(m)}_{,xxx} - 3\left(\frac{m\pi}{b}\right)u^{(m)}_{,xx}\right]$$
$$- B_{26}\left[\left(\frac{m\pi}{b}\right)^3 u^{(m)} - 3\left(\frac{m\pi}{b}\right)^2 u^{(m)}_{,x}\right] = p^{(m)}(x). \quad (33)$$

The general solution of the eight-order system (31)–(33) can be written as

$$u^{(m)}(x) = u^{(m)}_H(x) + u^{(m)}_P(x) \quad (34)$$

$$v^{(m)}(x) = v^{(m)}_H(x) + v^{(m)}_P(x) \quad (35)$$

$$w^{(m)}(x) = w^{(m)}_H(x) + w^{(m)}_P(x) \quad (36)$$

where $u^{(m)}_P$, $v^{(m)}_P$, $w^{(m)}_P$ are particular solutions and $u^{(m)}_H$, $v^{(m)}_H$ and $w^{(m)}_H$ are homogeneous solutions. Eliminating $u^{(m)}_H$ and $v^{(m)}_H$ from the homogeneous form of equations (31)–(33) an eighth-order ordinary differential equation with constant coefficients for $w^{(m)}_H$ is obtained as follows

$$(A_{11}A_{66}D_{11} - A_{11}B_{16}^2)\frac{d^8}{dx^8}w^{(m)}_H + \left(\frac{m\pi}{b}\right)^2(-2A_{11}A_{66}D_{12} - 4A_{11}A_{66}D_{66} - A_{11}A_{22}D_{11}$$
$$-6A_{12}B_{16}^2 + 4A_{66}B_{16}^2 + A_{12}^2D_{11} + 2A_{12}A_{66}D_{11} + 6A_{11}B_{16}B_{26})\frac{d^6}{dx^6}w^{(m)}_H$$
$$+\left(\frac{m\pi}{b}\right)^4(A_{11}A_{66}D_{22} + 2A_{11}A_{22}D_{12} + 4A_{11}A_{22}D_{66} + A_{22}A_{66}D_{11} + 20A_{12}B_{16}B_{26}$$
$$+8A_{66}B_{16}B_{26} - 9A_{22}B_{26}^2 - 2A_{12}^2D_{12} - 4A_{12}A_{66}D_{12} - 4A_{12}^2D_{66} - 8A_{12}A_{66}D_{66}$$
$$-9A_{11}B_{26}^2)\frac{d^4}{dx^4}w^{(m)}_H + \left(\frac{m\pi}{b}\right)^6(-A_{11}A_{22}D_{22} - 2A_{22}A_{66}D_{12} - 4A_{22}A_{66}D_{66} - 6A_{12}B_{26}^2$$
$$+4A_{66}B_{26}^2 + 6A_{22}B_{16}B_{26} + A_{12}^2D_{22} + 2A_{12}A_{66}D_{22})\frac{d^2}{dx^2}w^{(m)}_H + \left(\frac{m\pi}{b}\right)^8(A_{22}A_{66}D_{22}$$
$$-A_{22}B_{26}^2)w^{(m)}_H = 0. \quad (37)$$

The solution of (37) is of the form

$$w_H^{(m)}(x) = \sum_{k=1}^{8} c_k^{(m)} \mathrm{e}^{\lambda_k^{(m)} x} \tag{38}$$

where the $c_k^{(m)}$ ($k = 1$–8) are the arbitrary constants. The $\lambda_k^{(m)}$ ($k = 1$–8) are determined from the roots of the following auxiliary equation

$$\begin{aligned}
&(A_{11}A_{66}D_{11} - A_{11}B_{16}^2)\lambda^{(m)^8} + \left(\frac{m\pi}{b}\right)^2(-2A_{11}A_{66}D_{12} - 4A_{11}A_{66}D_{66} - A_{11}A_{22}D_{11} \\
&\quad - 6A_{12}B_{16}^2 + A_{12}^2D_{11} + 2A_{12}A_{66}D_{11} + 6A_{11}B_{16}B_{26})\lambda^{(m)^6} + \left(\frac{m\pi}{b}\right)^4(A_{11}A_{66}D_{22} \\
&\quad + 2A_{11}A_{22}D_{12} + 4A_{11}A_{22}D_{66} + A_{22}A_{66}D_{11} + 20A_{12}B_{16}B_{26} + 8A_{66}B_{16}B_{26} - 9A_{22}B_{26}^2 \\
&\quad - 2A_{12}^2D_{12} - 4A_{12}A_{66}D_{12} - 4A_{12}^2D_{66} - 8A_{12}A_{66}D_{66} - 9A_{11}B_{26}^2)\lambda^{(m)^4} \\
&\quad + \left(\frac{m\pi}{b}\right)^6(-A_{11}A_{22}D_{22} - 2A_{22}A_{66}D_{12} - 4A_{22}A_{66}D_{22} - 6A_{12}B_{26}^2 + 4A_{66}B_{26}^2 \\
&\quad + 6A_{22}B_{16}B_{26} + A_{12}^2D_{22} + 2A_{12}A_{66}D_{22})\lambda^{(m)^2} + \left(\frac{m\pi}{b}\right)^8(A_{22}A_{66}D_{22} - A_{22}B_{26}^2) = 0.
\end{aligned} \tag{39}$$

The Newton–Raphson method [13] can be used to determine the roots of the auxiliary equation (39). The numerical results show that for most of the engineering fiber-reinforced composite materials, there are four real roots and four complex roots for $\lambda^{(m)}$. Two of the four real roots are positive and the other two are negative. Of the complex roots, two have positive real parts and the other two have negative real parts.

For the case of 45° angle-ply composites, the closed form solution for auxiliary equation (39) can be obtained as follows

$$\lambda_{1,2,3,4}^{(m)} = \pm\frac{m\pi}{2b}\left[\left(\frac{\alpha}{2} + \sqrt{\left(\frac{\alpha}{2}\right)^2 + 2 - \beta}\right) \pm \sqrt{\left(\frac{\alpha}{2} + \sqrt{\left(\frac{\alpha}{2}\right)^2 + 2 - \beta}\right)^2 - 4}\right] \tag{40}$$

$$\lambda_{5,6,7,8}^{(m)} = \pm\frac{m\pi}{b}\left[\left(-\frac{\alpha}{2} + \sqrt{\left(\frac{\alpha}{2}\right)^2 + 2 - \beta}\right) \pm \sqrt{\left(-\frac{\alpha}{2} + \sqrt{\left(\frac{\alpha}{2}\right)^2 + 2 - \beta}\right)^2 - 4}\right] \tag{41}$$

where

$$\begin{aligned}
\alpha = {} & (A_{11}A_{66}D_{12} + 4A_{11}A_{66}D_{66} + A_{11}A_{22}D_{11} + 6A_{12}B_{12}^2 - 4A_{66}B_{16}^2 - A_{12}^2D_{11} \\
& - 2A_{12}A_{66}D_{11} - 6A_{11}B_{16}^2)/(A_{11}A_{66}D_{11} - A_{11}B_{16}^2)
\end{aligned} \tag{42}$$

$$\begin{aligned}
\beta = {} & (A_{11}A_{66}D_{22} + 2A_{11}A_{22}D_{12} + 4A_{11}A_{22}D_{66} + A_{22}A_{66}D_{11} + 20A_{12}B^2{}_{16} + 8A_{66}B_{16}^2 \\
& - 9A_{22}B_{16}^2 - 2A_{12}^2D_{12} - 4A_{12}A_{66}D_{12} - 4A_{12}^2D_{66} - 8A_{12}A_{66}D_{66} \\
& - 9A_{11}B_{16}^2)/(A_{11}A_{66}D_{11} - A_{11}B_{16}^2).
\end{aligned} \tag{43}$$

The numerical values of $\lambda_k^{(m)}$ obtained from equation (39) by the Newton–Raphson method are compared with the exact solutions (40) and (41) for 45° angle-ply laminates. It is found that the accuracy of the solution obtained by the numerical technique is very satisfactory. This justifies the use of the Newton–Raphson method to obtain the numerical solution of equation (39).

The in-plane homogeneous functions $u_H^{(m)}(x)$ and $v_H^{(m)}(x)$ are of the form

$$u_H^{(m)}(x) = \sum_{k=1}^{8} d_k^{(m)} e^{\lambda_k^{(m)} x} \tag{44}$$

$$v_H^{(m)}(x) = \sum_{k=1}^{8} f_k^{(m)} e^{\lambda_k^{(m)} x} \tag{45}$$

where $d_k^{(m)}$ and $f_k^{(m)}$ are constants which can be expressed in terms of $c_k^{(m)}$. For real values of $\lambda_k^{(m)}$, $d_k^{(m)}$ and $f_k^{(m)}$ are in the form

$$d_k^{(m)} = \{-[(A_{12}B_{16}-2B_{16}A_{66})\lambda_k^{(m)^4}+(3B_{16}A_{22}+3B_{26}A_{12}-2B_{26}A_{66})\lambda_k^{(m)^2}-B_{26}A_{22}]/$$

$$[A_{11}A_{66}\lambda_k^{(m)^4}+(A_{12}^2+2A_{12}A_{66}-A_{22}A_{11})\lambda_k^{(m)^2}+A_{22}A_{66}]\}\left(\frac{m\pi}{b}\right)c_k^{(m)} \tag{46}$$

$$f_k^{(m)} = \{[(A_{11}B_{16})\lambda_k^{(m)^4}+(3A_{12}B_{16}+2A_{66}B_{16})\lambda_k^{(m)^3}-(3A_{11}B_{26})\lambda_k^{(m)^2}$$
$$-(B_{26}A_{12}-2B_{26}A_{66})\lambda_k^{(m)}]/[A_{11}A_{66}\lambda_k^{(m)^4}+(A_{12}^2+2A_{12}A_{66}-A_{22}A_{11})\lambda_k^{(m)^2}$$
$$+A_{22}A_{66}]\}\left(\frac{m\pi}{b}\right)c_k^{(m)}. \tag{47}$$

For complex values of $\lambda_k^{(m)}$, $d_k^{(m)}$ and $f_k^{(m)}$ can also be expressed in terms $c_k^{(m)}$ but the forms are more complicated and lengthy and will not be given here. The displacements, strains, stresses and stress resultants can now be given in terms of eight arbitrary constants for each expansion term.

The eight independent arbitrary constants $c_k^{(m)}$ (k = 1–8) are to be determined from the eight remaining boundary conditions at $x = 0$ and $x = a$. For rectangular plates with semi-infinite length the boundary conditions at $x = a$ are replaced by regularity conditions, $u^{(m)}$, $v^{(m)}$, $w^{(m)}$ and $w_{,x}^{(m)}$ to be finite as $x \to \infty$.

It is noted that the ratio of in-plane displacements to transverse deflection, which is an index of the degree of coupling between stretching and bending, can be estimated from equations (46) and (47) before the entire problem is solved.

NUMERICAL EXAMPLES AND DISCUSSION

Finite plate under uniform load

Consider a finite plate under uniformly distributed transverse loading $p = p_0$ with simple support conditions (26) at $y = 0$ and $y = b$ and the following edge conditions at $x = 0$ and $x = a$:

Simply supported edges†

$$u_0 = 0$$
$$N_{xy} = 0$$
$$w^0 = 0 \tag{48}$$
$$M_x = 0.$$

† This corresponds to the problem considered by Whitney [7].

Clamped edges

$$\left.\begin{aligned} u^0 &= 0 \\ v^0 &= 0 \\ w^0 &= 0 \\ w^0_{,x} &= 0. \end{aligned}\right. \tag{49}$$

Free edges

$$\left.\begin{aligned} N_x &= 0 \\ N_{xy} &= 0 \\ M_x &= 0 \\ Q_x + M_{xy,y} &= 0 \end{aligned}\right. \tag{50}$$

where $Q_x + M_{xy,y}$ is the Kirchhoff shear force.

The coefficient $p^{(m)}$ of the half range Fourier series expansion (27) for the uniform transverse load p_0 is

$$u_P^{(m)} = \begin{cases} \dfrac{4p_0}{\pi m} & \text{for } m \text{ even} \\ 0 & \text{for } m \text{ odd.} \end{cases} \tag{51}$$

A set of particular solutions $u_P^{(m)}$, $v_P^{(m)}$ and $w_P^{(m)}$ are of the form

$$w_P^{(m)} = \begin{cases} \dfrac{4A_{66}b^4p_0}{(A_{66}D_{22}-B_{26}^2)(m\pi)^5} & \text{for even } m \\ 0 & \text{for odd } m \end{cases} \tag{52}$$

$$u_P^{(m)} = \begin{cases} \dfrac{B_{26}b^3p_0}{(A_{66}D_{22}-B_{26})(m\pi)^4} & \text{for even } m \\ 0 & \text{for odd } m \end{cases} \tag{53}$$

$$v_P^{(m)} = 0 \quad \text{for all } m. \tag{54}$$

In each layer let us denote the tensile moduli parallel and transverse to the filament direction by E_L and E_T, respectively, and the shear modulus by G_{LT} $(= E_T)$. The major Poisson's ratio ν_{LT} is assumed to be 0·25. The total number of layers is denoted by n $(= 2, 4, 6, \infty)$. The case $n = \infty$ actually denotes the solution which neglects the coupling terms (i.e. $B_{16} = B_{26} = 0$) and will be called the "uncoupled" solution.

The center deflection of n-layer rectangular plates is plotted vs. angle θ in Figs. 1–3 for the various boundary conditions at $x = 0$ and $x = a$ given by (48)–(50). Figure 4 gives the variation of center deflection with the aspect ratio a/b for $\theta = 30°$. The degree of anisotropy E_L/E_T is taken to be 40 which is typical of a graphite–epoxy composite. For large number of layers ($n \geq 6$ in the present case) the "uncoupled" solution is a good approximation for the center deflection. It can be seen that the coupling effect becomes very significant for two-layer laminates with large angle θ. This phenomena was first discussed by Whitney and Leissa [6]. The effect of in-plane boundary conditions is shown in Tables 1 and 2 for

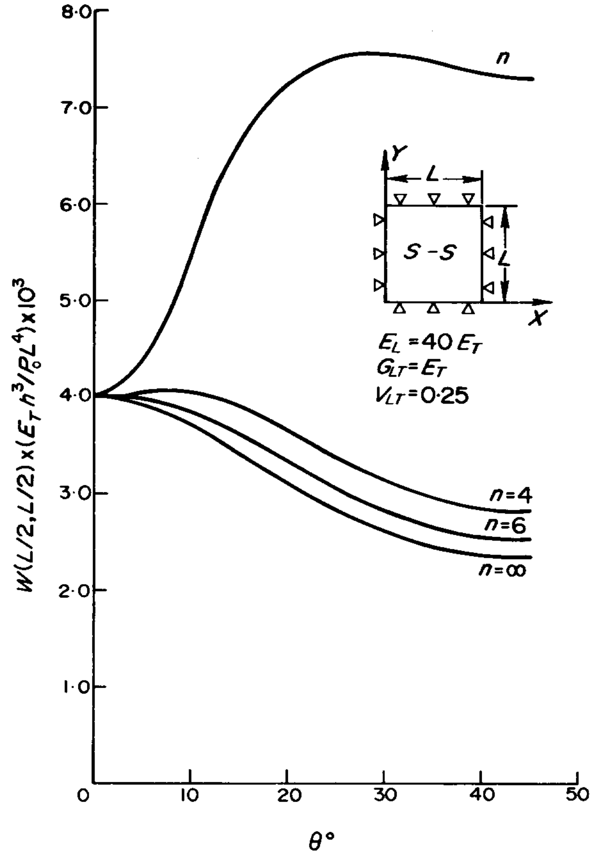

FIG. 1. Center deflection of angle-ply square plates under uniform load with simple support–simple support (*S–S*) edges vs orientation angle.

various bending boundary conditions and different angle θ. It can be seen from Tables 1 and 2 that the type of in-plane boundary condition is a significant factor for simply supported but not clamped plates.

These results are based on a five term single Fourier series solution. For comparison Table 1 also includes, in brackets, the double Fourier series solution of Whitney [7] with 121 terms for simply supported edges. It is seen that the present method requires only a few terms in the expansion in order to obtain relatively accurate results.

Plates with infinite length under uniform line load

Consider a simply supported rectangular plate of infinite length. The simply supported conditions are given by equations (26). The width of the plate is b. It is uniformly loaded along the y-axis, as shown in Fig. 5.

The uniformly distributed line load along the y-axis can be represented by the following expansion

$$P = P_0 = P_0 \frac{4}{\pi} \sum_{m=1,3,5}^{\infty} \frac{1}{m} \sin \frac{m\pi y}{b} \tag{55}$$

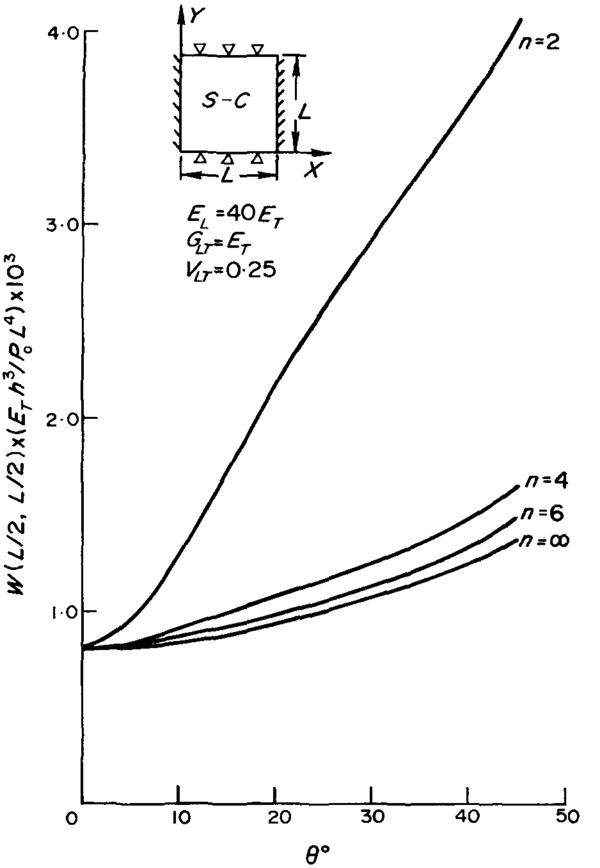

FIG. 2. Center deflection of angle-ply square plates under uniform load with simple support–clamped (S–C) edges vs orientation angle.

where P_0 is the uniform load/unit length. From the condition of symmetry, it follows that at $x = 0$

$$\begin{aligned} Q_x &= \frac{P_0}{2} = -\frac{2}{\pi}P_0 \sum_{m=1,3,5}^{\infty} \frac{1}{m}\sin\frac{m\pi y}{b} \\ w^0_{,x} &= 0 \\ u^0 &= 0 \\ v^0 &= 0. \end{aligned} \tag{56}$$

The regularity conditions give that u^0, v^0, w^0 and $w^0_{,x}$ are finite as x approaches infinity.

The deflection along the center line $y = b/2$ for two and four-ply and "uncoupled" laminates with angle $\theta = 30°$ is shown in Fig. 5. It is clearly shown that the degree of coupling is severe for 30° laminates. Comparing these results with those of uniform load

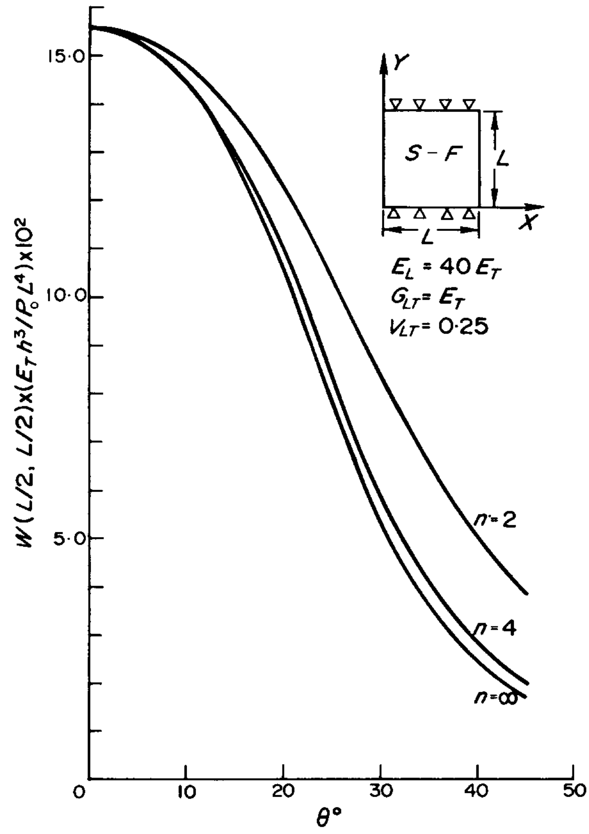

FIG. 3. Center deflection of angle-ply square plates under uniform load with simple support–free (*S–F*) edges vs orientation angle.

given in Fig. 4 for high aspect ratios, it is seen that the effect of coupling is dependent on the type of transverse loading. This contradicts the conclusion by Whitney [7] that the effect of coupling is independent of the type of transverse loading.

Plate with semi-infinite length under uniform edge moment

Consider a rectangular plate with semi-infinite length ($a \to \infty$) and simple support edge conditions (26) at $y = 0$ and $y = b$ under uniform edge moment M_0 as follows:

$$\left.\begin{aligned} u^0 &= 0 \\ v^0 &= 0 \\ w^0 &= 0 \\ M_x &= M_0 \end{aligned}\right\} \quad \text{at } x = 0 \tag{57}$$

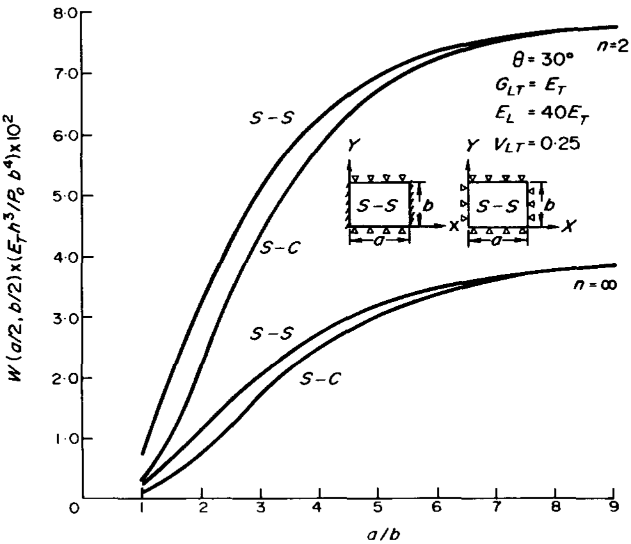

FIG. 4. Center deflection of 30° angle-ply plates under uniform load vs aspect ratio.

TABLE 1. CENTER DEFLECTION OF AN ANGLE-PLY SQUARE PLATE UNDER UNIFORM LOAD WITH SIMPLE SUPPORT—SIMPLE SUPPORT EDGES ($E_L/E_T = 40$, $G_{LT} = E_T$, $\nu_{LT} = 0{\cdot}25$)†

Boundary conditions $w^0 = M_y = N_{xy} = v^0 = 0$ at $y = 0, L$			$w^0\left(\frac{L}{2}, \frac{L}{2}\right) \times \frac{E_T h^3}{p_0 L^4} \times 10^2$			
Bending B. C. at $x = 0, L$	In-plane B. C. at $x = 0, L$	Angle θ (deg.)	$n = 2$	$n = 4$	$n = 6$	$n = \infty$
			[0·44395]	[0·40458]	[0·39804]	
$w^0 = M_x = 0$	$u^0 = N_{xy} = 0$	5	0·4439	0·4045	0·3980	0·3927
	$N_x = N_{xy} = 0$		0·4435	0·4045	0·3980	
	$u^0 = v^0 = 0$		0·4184	0·3988	0·3955	
	$N_x = v^0 = 0$		0·4164	0·3983	0·3953	
			[0·75765]	[0·31461]	[0·28387]	
$w^0 = M_x = 0$	$u^0 = N_{xy} = 0$	30	0·7576	0·3146	0·2838	0·2633
	$N_x = N_{xy} = 0$		0·7546	0·3144	0·2838	
	$u^0 = v^0 = 0$		0·6510	0·3084	0·2815	
	$N_x = v^0 = 0$		0·5957	0·3039	0·2798	
			[0·73368]	[0·28324]	[0·25433]	
$w^0 = M_x = 0$	$u^0 = N_{xy} = 0$	45	0·7337	0·2832	0·2543	0·2351
	$N_x = N_{xy} = 0$		0·7337	0·2832	0·2543	
	$u^0 = v^0 = 0$		0·6812	0·2809	0·2535	
	$N_x = v^0 = 0$		0·6508	0·2788	0·2528	

† Brackets enclose results based on double Fourier series method from Whitney [7].

TABLE 2. CENTER DEFLECTION OF AN ANGLE-PLY SQUARE PLATE UNDER UNIFORM LOAD WITH SIMPLE SUPPORT—CLAMPED EDGES ($E_L/E_T = 40$, $G_{LT} = E_T$, $\nu_{LT} = 0{\cdot}25$)

Boundary conditions $w^0 = M_y = N_{xy} = v^0 = 0$ at $y = 0, L$			$w^0\left(\frac{L}{2},\frac{L}{2}\right) \times \frac{E_T h^3}{p_0 L^4} \times 10^2$			
Bending B. C. at $x = 0, L$	In-plane B. C. at $x = 0, L$	Angle θ (deg)	$n = 2$	$n = 4$	$n = 6$	$n = \infty$
$w^0 = w^0_{,x} = 0$	$N_x = v^0 = 0$	5	0·0946	0·0837	0·0819	0·0806
	$N_x = N_{xy} = 0$		0·0946	0·0837	0·0819	
	$u^0 = v^0 = 0$		0·0943	0·0836	0·0819	
	$u^0 = N_{xy} = 0$		0·0942	0·0836	0·0819	
$w^0 = w^0_{,x} = 0$	$N_x = v^0 = 0$	30	0·2945	0·1262	0·1411	0·1060
	$N_x = N_{xy} = 0$		0·2950	0·1262	0·1411	
	$u^0 = v^0 = 0$		0·2946	0·1262	0·1411	
	$u^0 = N_{xy} = 0$		0·2942	0·1262	0·1411	
$w^0 = w^0_{,x} = 0$	$N_x = v^0 = 0$	45	0·4068	0·1650	0·1487	0·1377
	$N_x = N_{xy} = 0$		0·4068	0·1650	0·1487	
	$u^0 = v^0 = 0$		0·4068	0·1650	0·1487	
	$u^0 = N_{xy} = 0$		0·4068	0·1650	0·1487	

and

$$\left.\begin{aligned} |u^0| &< \infty \\ |v^0| &< \infty \\ |w^0| &< \infty \\ |w^0_{,x}| &< \infty \end{aligned}\right\} \text{as } x \to \infty. \tag{58}$$

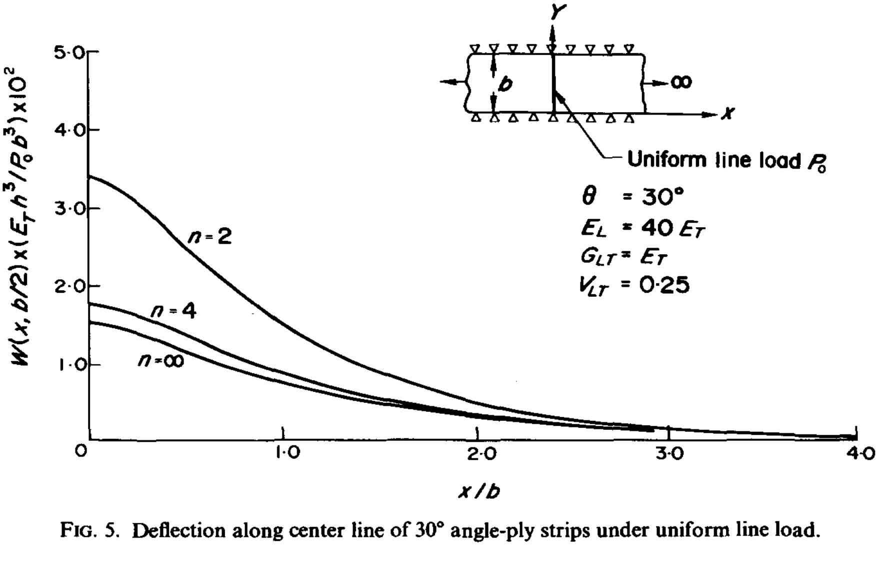

FIG. 5. Deflection along center line of 30° angle-ply strips under uniform line load.

There is no transverse normal loading p. The edge bending moment at $x = 0$ is expanded into the following Fourier series

$$M_x = M_0 = \frac{4M_0}{\pi} \sum_{m=1,3,5}^{\infty} \frac{1}{m} \sin \frac{m\pi y}{b} \tag{59}$$

to obtain the solution of this problem. The deflection along the center line ($y = b/2$) are shown in Fig. 6. The maximum deflection occurs at $x = 0{\cdot}50b$ for 30° angle-ply laminate.

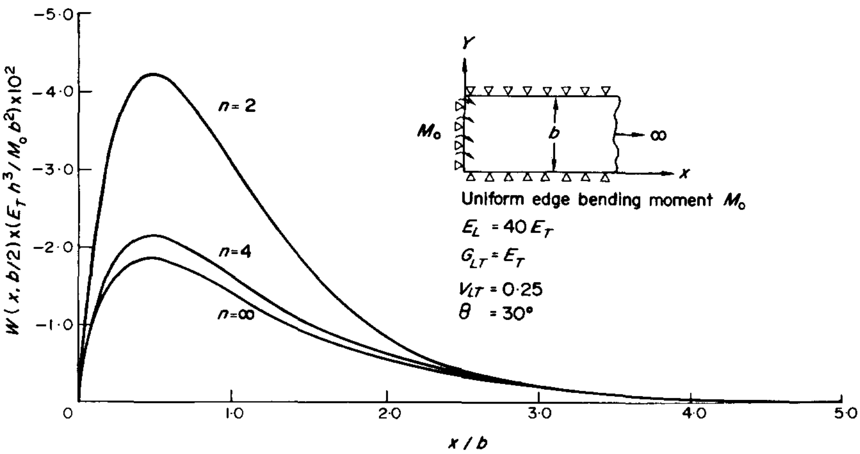

FIG. 6. Deflection along center line of 30° angle-ply strips under uniform edge bending moment.

CONCLUSIONS

The results of this study yield the following conclusions:

1. The degree of coupling does not only depend on the degree of anisotropy of material properties, the angle of orientation and the total number of layers in the laminate. It also depends on the type of transverse loading, the aspect ratio and the edge boundary conditions. The type of in-plane boundary condition (movable or immovable) has a significant effect for simply supported plates but not for clamped plates. The orientation angle has a very significant effect for two-layer laminates.

2. The ratios of the in-plane displacements to transverse deflection ($u^{(m)}/w^{(m)}$, $v^{(m)}/w^{(m)}$), which is an index of the degree of coupling, can be estimated before the entire problem is solved, as indicated in equations (46) and (47).

3. The present single Fourier series solution for unbalanced angle-ply laminates coverges more rapidly than previous double Fourier series solutions [7–9]. It is more suitable to use the present single series solution for numerical computation especially if higher derivates of the displacements u^0, v^0 and w^0 are involved. This has been discussed in detail by Timoshenko [14] for the Navier and Levy solutions for simply supported homogeneous rectangular plates.

4. The present method can solve a wide class of problems in geometry (i.e. rectangular plates with finite, semi-infinite and infinite length), as well as boundary conditions (only two opposite simply supported edges are required; the remaining edge boundary conditions can be arbitrary). It can be used with the method of superposition to solve problems having other edge conditions as well as transverse loading conditions.

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Абстракт—Исследуются неуравновесные, слоистые, со слоями наклоненными под углом, прямоугольные пластинки под влиянием поперечной нормальной нагрузки. Путем разложения нагрузки и перемещений в ординарные ряды фурье, определяющие дифференциальные уравнения в частных производных сводятся к обыкновенным дифференциальным уравнениям для простих граничных условый операция на двух противополюженных краях и произвольных граничных условиях на двух других краях. Получается решение в замкнутом виде для обыкновенных дифференциальных уравнений, для угла наклона слоев равного 45°. Даются некоторые численные примеры результаты которых представляются в виде графиков. Порядок сопряжения между расширением и изгибом зависит от поперечной нагрузки, граничных краевих условий, аспекта отношения пластинки, а также от порядка анизотропии слоев, ориентации угла и числа слоев. Тип граничного условия в плоскости /передвижный или нет/ имеет значительный эффект на изгиб свободно опертых пластинок, но не для защемленных. Обсуждаемый в работе метод рядов Фурье можно использовать, вместе с методом суперпозиции, для расчёта задач обладающих другими краевыми условиями, а также другими условиями поперечной нагрузки.